Introduction to Rotational Motion

Disclaimer: This handout is thought of mainly as a recap of Newton's laws of rotational motion, for completeness. If you are already familiar with these, you may skip to the next handout going through olympiad techniques (although you may find the "Exploring the deep" problems interesting here). And in fact, the recap here is very bare bones, and if you have never heard of rotational motion before I would encourage a more proper explanation in any of the clasical mechanics textbook I list on my site. If you are familiar but need a refresher, this handout is for you.

1 Where are we now?

So, we brought order to the universe by introducing Newton's laws, which let us describe a lot of the motion (and lack thereof) that we see in the Universe. But, our picture appears to be incomplete. Consider this case:

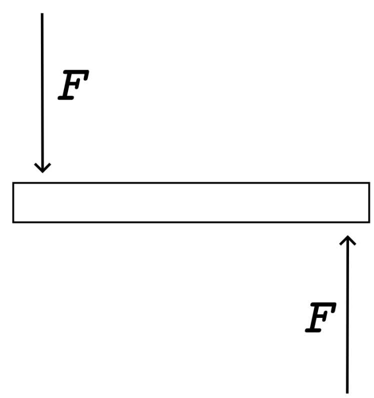

Figure 1: Beam with opposing forces, what will happen?

The object has one force F acting in the positive vertical direction. And one in the negative vertical direction. The forces cancel, and hence the system is in equilibrium by Newton's first law. However, is that the full picture?

Think for yourself: What will happen to the block?

1.1 Rotation

The answer to the question above is that we need to revise - or rather, clarify - Newton's laws somewhat. The acceleration that Newton's laws talk about is the acceleration of the *centre of mass of* an object. For completeness, the centre of mass is calculated as the weighted mean of the position of mass across the object:

$$x_{com} = \sum_i m_i x_i$$

In this way, Newton's laws tell you everything there is to know about *translational* motion of the centre of mass. However, motion can still occur *around* the centre of mass - rotational motion. This will happen if there is an asymmetry of forces around the center of mass. In the case above, that is exactly what is happening. There is no net force on the object as a whole, hence the centre of mass will be still. But there is an asymmetric force around the centre of mass, which will give rise to rotation (in the counter-clockwise direction).

Now, let's get more precise about all of this, and specifically write down the Newton equivalent laws for rotational motion. Like in the case of Newton's laws, I will assume that you are already familiar with these, and hence just recall them here, with no further explanation. See (Halliday & Resnick) or any other suitable high school/first year university textbook for a fuller explanation.

1.2 Newton's laws for rotation

Newton's laws for translational motion are centred around the *force* and the *mass*. The force gives the size of disruption to a system, and the mass tells you how unwilling the system is to be disrupted. These have direct equivalents in rotational motion, in *torque* and *moment of inertia*. Torque τ and moment of inertia I are defined in the following ways:

$$\tau = \boldsymbol{r} \times \boldsymbol{F}$$
$$I = \sum_i m_i x_i^2$$

For torque τ, $\boldsymbol{F}$ is the applied force, and $\boldsymbol{r}$ is the position vector of the point of application of the force. This leads to the question, position with respect to wqhat origin? We will delve into this question a little bit later, but for now you may consider the point of rotation as the point of origin. If you have never seen the cross product between two vectors before, don't worry! For now, you can consider this as just taking $Fr\sin\theta$ where θ is the angle between the force and the position vector. For the moment of inertia I, m_i symbolizes all the mass elements making up the object, and x_i their respective distances to the axis of rotation.

Now we have defined the basics. They may feel a little arbitrary for now, especially if you have never seen them before (that is good! You should be asking yourself why the laws are this way, eventually you will learn, but it will take some time to get there:)). In any case, now that we have introduced these quantities, we can formulate Newton's equivalent laws for rotational motion (I have highlight the words which are different from the formulation of Newton's translational laws of motion in our previous handout:

I. A body remains at rest or moving with constant *angular* velocity, if and only if there is no net *torque* acting on it.

II. The net *torque* on a body equals the body's *angular* acceleration times its *moment of inertia* ($\tau = I\alpha$)

III. If one body exerts a *torque* on a second body, then that second body always exerts a *torque* equal in magnitude but opposite in direction on the first body.

To really map it out, here is a table to clarify things:

Translational (units)	**Rotational (units)**
Force (N)	Torque (Nm)
Mass (kg)	Moment of Intertia (kg m^2
Displacement (m)	Angular displacement (rad)
Velocity (m/s)	Angular velocity (rad/s)
Acceleration (m/s^2)	Angular acceleration (rad/s^2)
Momentum (kg m/s)	Angular momentum (kg m^2/s)

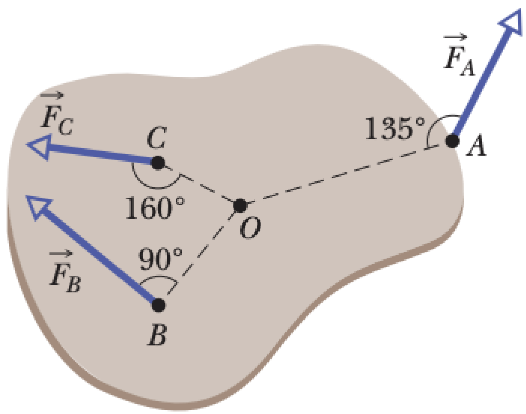

Figure 2: Torques on object

2 Questions

2.1 Check your understanding

1. What is the angular speed of (a) the second hand, (b) the minute hand, and (c) the hour hand of a smoothly running analog watch? Answer in radians per second. (Halliday & Resnick)

2. The length of a bicycle pedal arm is 0.152 m, and a downward force of 111 N is applied to the pedal by the rider.What is the magnitude of the torque about the pedal arm's pivot when the arm is at angle (a) 30°, (b) 90°, and (c) 180° with the vertical? (Halliday & Resnick)

3. The body in Fig. 2 is pivoted at O. Three forces act on it: $F_A = 10$ N at point A, 8.0 m from O; $F_B = 16$ N at B, 4.0m from O; and $F_C = 19$ N at C, 3.0m from O. What is the net torque about O? (Halliday & Resnick)

4. If a 32.0 Nm torque on a wheel causes angular acceleration 25.0 rad/s^2, what is the wheel's rotational inertia? (Halliday & Resnick)

2.2 Trying out the waters

5. A cylinder of mass M, radius r, is set to rotate with angular velocity ω_0 about its own axis, which is fixed. If after time t the angular velocity is ω_1, find the frictional torque N on the cylinder at time t, assuming that N is constant. (Oxford)

6. A ladder leans against a frictionless wall while the bottom rests on a horizontal floor with coefficient of friction μ as shownin Fig. 4. Show that the smallest angle that the ladder can make with the floor without slipping is given by $\tan\theta_{min} = 1/2\mu$. (Oxford)

7. A light thread with a body of mass m tied to its end is wound on a uniform solid cylinder of mass M and radius R (Fig. 3). At a moment $t = 0$ the system is set in motion. Assuming the friction in the axle of the cylinder to be negligible, find the time dependence of the angular velocity of the cylinder. (The moment of inertia of a cylinder around its central axis is $\frac{1}{2}MR^2$(Irodov)

2.3 Explore the deep

Some maths...

8. Calculate the moment of inertia of a uniform thin rod of mass M and length L around its a) endpoint b) centre of mass. *Hint*: you may wish to integrate over all the small dm to get the moment of inertia.

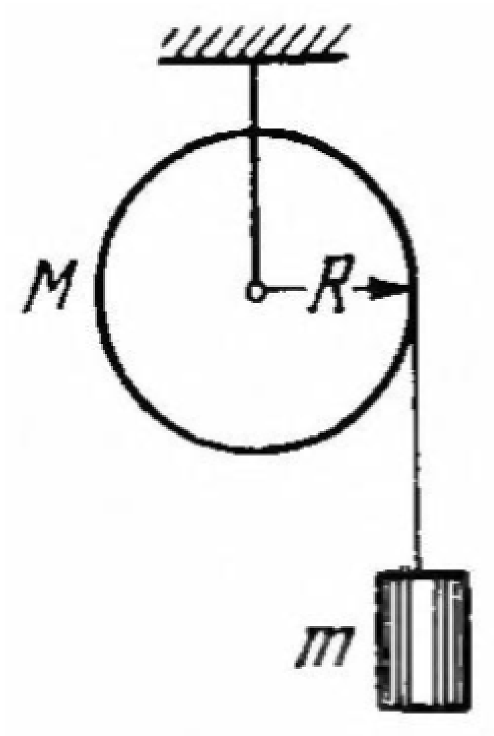

Figure 3: Mass attached to cylinder

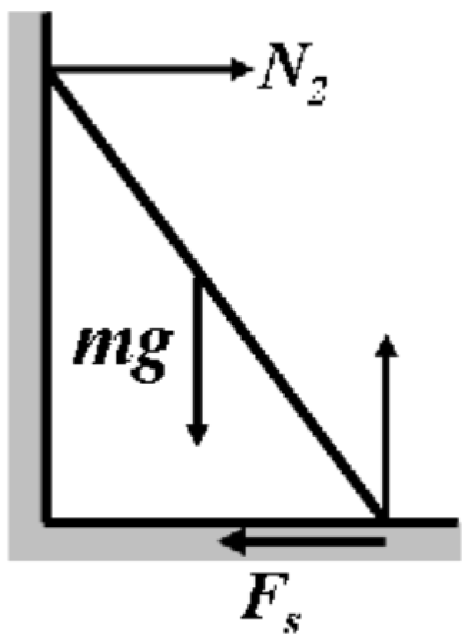

Figure 4: Ladder on wall

9. Calculate the moment of inertia of a uniform cylinder of mass M, radius R and length L around its central axis.

10. Calculate the moment of inertia of a uniform sphere of mass M, and radius R around any axis through the sphere.

...and some physics!

11. A uniform disc of radius R is spinned to the angular velocity ω and then carefully placed on a horizontal surface. How long will the disc be rotating on the surface if the friction coefficient is equal to μ? The pressure exerted by the disc on the surface can be regarded as uniform. (Irodov)

12. In the arrangement shown in Fig. 5 the mass of the uniform solid cylinder of radius R is equal to m and the masses of two bodies are equal to m_1 and m_2. The thread slipping and the friction in the axle of the cylinder are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions T_1/T_2 of the vertical sections of the thread in the process of motion. *Hint*: Tension force goes toward acceleration the mass at the end of the rope, but also toward rotating the cylinder. (Irodov)

13. A plate, bent at right angles along its center line, is placed on a horizontal fixed cylinder of radius R as shown in Fig. 6. How large does the coefficient of static friction between the cylinder and plate need to be if the plate is not to slip off the cylinder? (PPP 44, through Kevin Zhou)

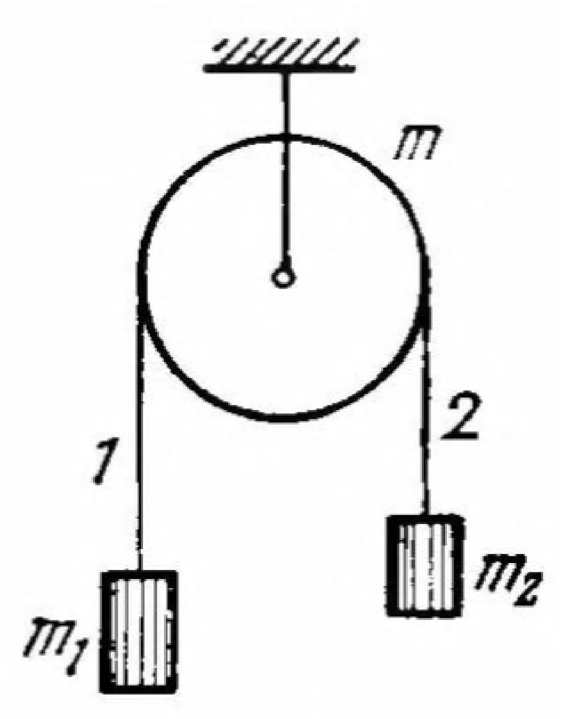

Figure 5: Pulley with friction

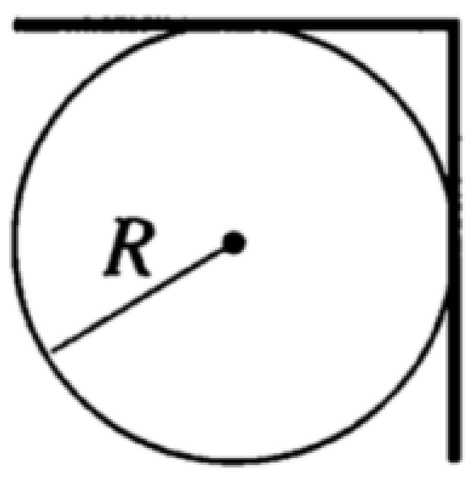

Figure 6: Plate on cylinder