Geometric Force Techniques

1 Where are we now?

In the last handout we learned about the fundamental laws governing mechanics, as codified in Newton's Laws of Mechanics. At the heart of Newton's laws is the *force*, a concept which Newton introduced. We solved a variety of problems with forces in the last handout, getting used to how they work. Today, we will learn a specific technique that we can apply when working with forces in order to ease problem solving, namely the *geometric technique*.

2 Geometric Force Techniques

2.1 The Prince and the Pauper

For many problems in physics, mathematics and elsewhere, there exist two kinds of solution, the *prince's* way, and the *pauper's* way. Consider the follwoing set of equations:

$$\begin{aligned} a + 3b - 2c &= 3 \\ -2a + b + 3c &= 2 \\ 3a - 2b + c &= 1 \end{aligned}$$

If I ask you to find the sum $a + b + c$, how would you do it? Of course, since this is a system of three equations with three unknowns, you could solve the system individually for a, b, and c, and then sum them together. This would give the right answer (barring any algebraic mistakes), but would be quite work intensive (especially for larger systems of equations). This is the *pauper's* way, reliable but tedious. The *prince's* way is oftentimes more difficult to find, and rests on leveraging some trick in the question. The first thing to realize is that the problem only asks for the sum of the three numbers, and not their individual values. Hence, when we find all the values, we seem to be doing more work than the question is asking for. This is a clue that there is a royal path available to us. I will leave it up to you to find it![1]

2.2 The Prince of Mechanics

Finding the *prince's* way is oftentimes tricky, and sometimes not even feasible. However, it can be very rewarding. Today, we will learn one technique which will help us solve a host of physics problems in the prince's way. You will most likely be used to solving these problems in the pauper's way, and may initially find this new approach dificult at first. This is very natural. I encourage you to really stick with this approach throughout this handout, and hopefully after a few problems you will get the hang of it. Without further ado, let's dig in.

To introduce this technique, the very first thing to realize is that a force is a vector. This is very significant. A vector is a quantity with both a direction and a magnitude. We often choose to represent this as an arrow, where the length of the arrow shows the magnitude. This is very powerful.

However, you will likely have seen from the last handout that we can project a force vector onto a dimension, and then only consider the component of the force along a certain direction (oftentimes the direction along a slope or another surface). This is a very useful trick, and works wonders in many situations,

[1] Hint: See how many times each unkown appears in all the equations.

especially when we are interested in knowing the forces only along certain directions. But, this can also lead to a lot of messy algebra, and we should always try to avoid making any unnecessary projections before considering the geometry of the problem in front of us - projections which do not respect the innate geometry and symmetry of the mechanical system we are considering will be tedious and will not give us any valuable insights. And so, there are many cases when we can get away with making no projections, and thus no algebraic manipulations, working only with the force vectors in their vector form (as arrows). This can often save us the pain of doing a lot of boring algebra, and can lead to very elegant solutions. This is at the heart of *geometric* solutions to statics problems.

This is particularly useful in problems involving statics, where a system is still. Additionally, this is particularly useful where friction is involved, as you will see below. If you find yourself in front of a statics problem with friction, you may save yourself a lot of time (and impress your friends) by attempting a geometric solution.

2.3 Example: Moving a block

This technique is perhaps most easily explained through an example. Let's consider this question:

> A person pulls on a block with a force F, at an angle of θ with respect to the horizontal. The coefficient of friction between the block and the ground is μ. For what θ is the F required to make the block slip a minimum? What is the corresponding F? (Morin)

You will likely be able to solve this in the typical way, projecting forces onto axes, and then minimizing the magnitude of F.

Task 1: Solve the problem above.

I will now show how to solve this problem geometrically. First, we note that the only forces acting on the block is the graivational force (weight) W, the force F from the person, the normal force N, and the friction force $F_d = \mu N$. We will sum all these forces vectorially and then analyze.

2.3.1 Building an intuition

We start with the only force we can determine with certainty, the weight. We know for a fact that it is of a magnitude mg in the negative y-direction (see figure). Then, to gain an intuition (this part is not actually not strictly necessary to solving the problem), we draw the force which we can vary, F, and add it to the weight force vector. Recall that to add one vector to another we simply place the base of one of the vectors by the tip of the other, the sum is then the vector from the free base of the latter vector to the free tip of the former).

Lastly, we turn to the *derived forces*. In this case, this will be the normal force and the friction force. I refer to them as derived forces because their value will depend on the other forces in the system. For friction, this is most clearly the case, since $F_d = \mu N$, and so it depends on the normal force. But it is important to understand that the normal force will also be derived, as it will adjust so as to balance forces on the block in the y-direction (I encourage you to take a moment to consider this). We start with drawing the normal force and adding it to our mechanical system. We know that the normal force will be normal to the surface (always be grateful when naming in physics is appropriate[2].), and hence will be in the positive y-direction. Secondly, we know that it will work to balance the net force in the y-direction, and thus it will have its tip at the same y coordinate as the base of the weight force vector. Finally, we add the friction force. This one will be perpendicular to the normal force (in the x-direction, opposite to the projection of the force F on the x-axis), and of magnitude μN.

Now that we have a final configuration, we can reason about it. What condition will our force configuration fulfill for the object to move? Think aobut this for yourself first.

Question: What condition must the force configuration fulfill for the object to move?

[2] See "electromotive force"

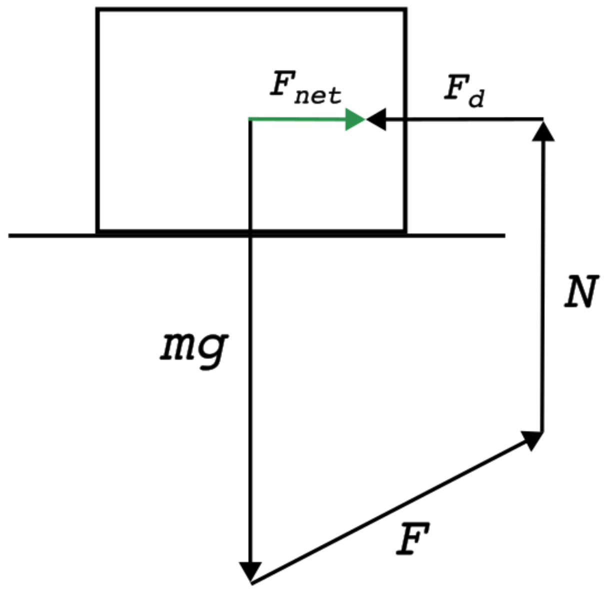

Figure 1: Geoemtric force configuration

The correct answer comes straight from Newton's first law - as is the case for essentially all questions in statics.[3] Namely, Newton's first law states that for the object to accelerate (i.e. start moving from its stationary position on the ground), the net force must be non-zero. Geometrically, a net zero corresponds exactly to our configuration closing onto itself, since the sum of all the vectors (which is the vector from the first vectors base to the last vectors tip) would then be the zero vector. So, for acceleration to occur and the object to slip the vector configuration must not close.

Now, however, we were looking for the minimal force required. At the minimal force, the object is *just barely* slipping, meaning that the net force is just infitesimally larger than the zero vector. And so, in the limiting case, the minimal force will be achieved when the force configuration is just precisely closing onto itself, and so we should in fact be looking for the case when the vector force configuration closes.[4]

2.3.2 Solving the problem

With this in hand, we can see that for the case we just went through above (see figure 2) the force applied is not minimal, as the object starts slipping with a larger than zero acceleration. To find the minimal slipping force, we will now do the problem backwards, adding the variable force in at last (and in fact, we could have done this to begin with, but hopefully the discussion above was informative).

This time, we once again start with the force we know, but after that, we start from the back. We know that we want the friction force vector to end exactly where the weight vector starts. We draw out on friction force vector at random. Now, since $F_d = \mu N$, we get $N = F_d/\mu$, and we also know that the normal force vector will be in the positive y direction. Thus, after ahving drawn out the friction force we uniquely determine the normal force, and we can now see exactly how large the variation force will have to be (and

[3] With the added corresponding Newton's first law for rotational equailibrium and torques.

[4] This is a very common trick in mechanics, where finding the minimal force required to break the equilibrium state amounts to finding the limiting case of the equilibrium state. This essentially turns the problem into one of statics, even though we are considering an object which will begin moving.

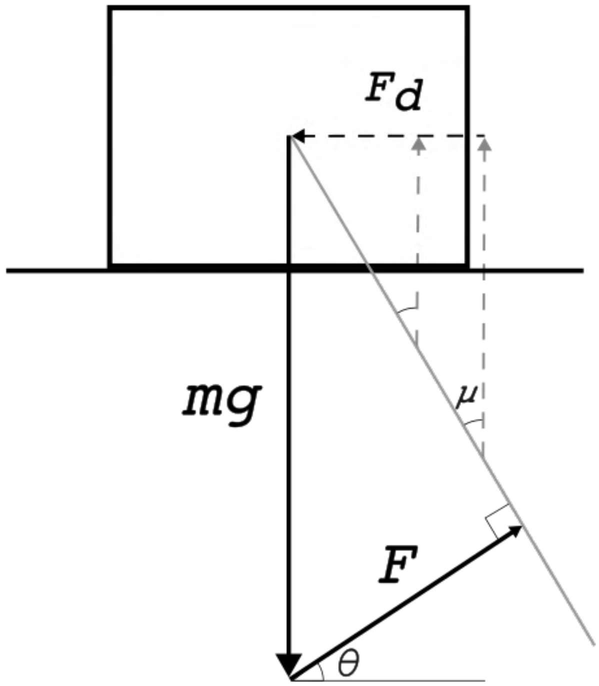

Figure 2: Finding the minimal force through geometry

in what direction) for the configuration to close. Great! However, we just picked a friction force vectore at random. How do we find the optimum? Ideally, we would like to draw out all the possible friction force vectors, and their corresponding normal force vectors, and see which combination gives rise to the smallest force F required to close the configuration. Let's do that.

To draw out all the possible friction, normal force combinations, we leverage an idea which is very central to this approach (and you will use in all the questions below). Namely that, since the friction force is orthogonal to the normal force, and their relative magnitude is always μ, all the possibilities of friction and normal force map out a straight line in space, with an angle $\arctan\mu$ to the normal (see figure **??**. And so, we can now draw out all the possible friction, normal force sums, and we end up with a straight line.

Now, the problem is simply one of finding the shortest distance between this line and the tip of the weight force vector. This will give us the minimum force required to close the configuration, and the corresponding angle θ will tell us the angle for which the force F is minimal. This is done by simple geometry (see figure **??**), where the most important thing to note is that the shortest distance is found where at the perpendicular to the line. We find that

$$\theta = \arctan\mu$$

Tada! This may seem like a slightly tedious method, but once you get the hang of it, I promise you that you will be able to solve some pretty tricky mechanics problems really fast (avoiding all kinds of tedious algebra). When we move onto rotational statics problems as well, the geometric approach will become even more powerful.

If you feel like you don't quiter understand how this works at the moment, don't worry! The best way to get a hang of this technique (as is the case for nearly everything in olympiads) is to try it yourself, and the first question below is to recreate this derivation for yourself. Below I have listed the steps I recommend to apply the geometric force technique to mechanics problems, and I encourage you to consult it when trying the problems below.

2.3.3 Method for solving problems geometrically

1. List all the forces acting on the object. Decide which forces are completely known (independent of other forces), and what forces are derived (i.e. will depend on the other forces).
2. Draw the known forces (oftentimes weight, always mg in the negative y-direction. If multiple, sum them together into one large vector (add vectors by putting the base of one at the tip of another).
3. Draw the variable forces (oftentimes some supplied external force, such as F in the problem above. Add this vector to the known force vector (again, add vectors by putting the base of one at the tip of another).
4. Lastly, draw the derived force vectors (oftentimes the normal force and the friction force).
5. **For static equilibrium, these forces should all add to zero**, meaning that the tip of the last force vector ends up at the base of the first force vector.
6. Finally, map out all the possible value for the derived forces (for the common case of normal force and friction, this will be a straight line). Finding the minimal variable force is now simply a problem of finding the minimal distance from the tip of the known force vector to the line of possible derived force vectors.

3 Questions

1. Solve the problem from the introduction on your own, using the geometric force technique.
2. A book of mass M is positioned against a vertical wall. The coefficient of friction between the book and the wall is μ. You wish to keep the book from falling by pushing on it with a force F applied at an angle θ with respect to the horizontal ($-\pi/2 < \theta < \pi/2$), as shown in figure 3. (Morin)

   a) For a given θ, what is the minimum F required?

   b) For what θ is this minimum F the smallest? What is the corresponding minimum F?

   c) What is the limiting value of θ, below which there does not exist an F that keeps the book up?
3. A bar of mass m is pull up by means of a thread up an inclined plane forming an angle α with the horizontal (see figure 4). The coefficient of friction is equal to k. Find the angle β which the thread must form with the inclined plane for the tension of the thread to be minimum. What is it equal to? (Irodov)
4. A wedge with the angle α at the tip is lying on the horizontal floor, see figure 5. There is a hole with smooth walls in the ceiling. A rod has been inserted snugly into that hole, and it can move up and down without friction, while its axis is fixed to be vertical. The rod is supported against the wedge; the only point with friction is the contact point of the wedge and the rod: the friction coefficient there is μ. For which values of μ is it possible to push the wedge through, behind the rod, by only applying a sufficiently large horizontal force? (Kalda)
5. What is the minimum force needed to dislodge a block of mass m resting on an inclined plane of slope angle α, if the coefficient of friction is μ? See figure 6. Investigate the cases when a) $\alpha = 0$; b) $0 < \alpha < \arctan \mu$. (Kalda)
6. The task of question 2 is to find the minimum force required to keep the book still. What is the maximum allowable force, as a function of θ and μ? Is there a speical angle that arises? Given μ, make a rough plot of the allowed values of F for $-\pi/2 < \theta < \pi/2$. (Morin)
7. A block of mass M is positioned underneath an overhang that makes an angle β with the horizontal. You apply a horizontal force of Mg on the block, as shown in figure 7. If the coefficient of static friction is μ, for what range of angles β does the block remain at rest? (Morin)

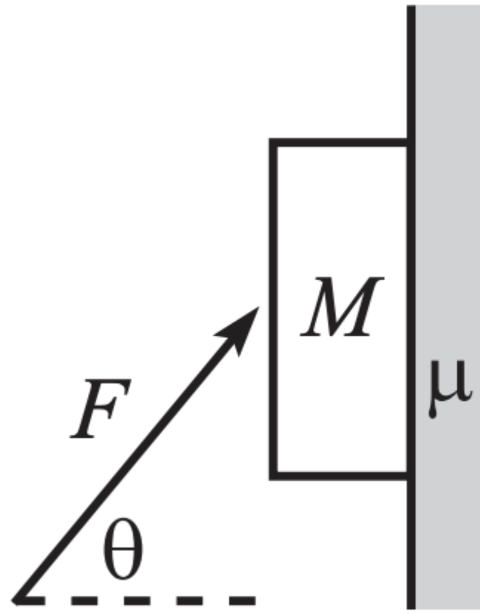

Figure 3: Book on wall

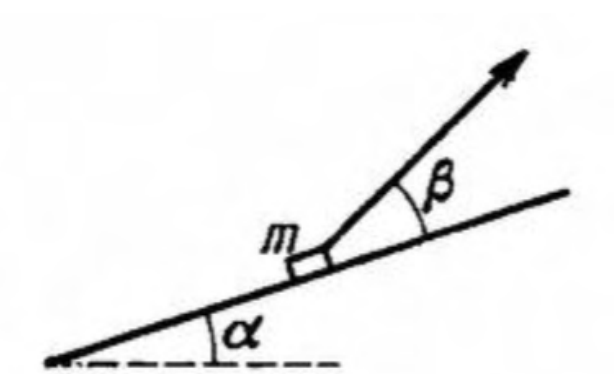

Figure 4: Mass pulled up on incline

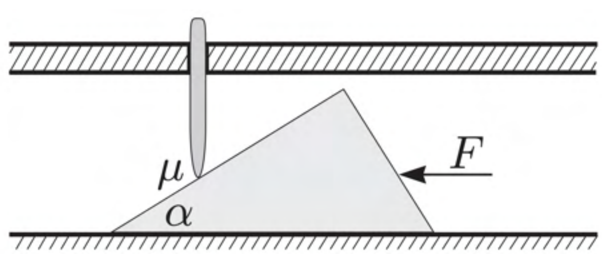

Figure 5: Rod stopping block

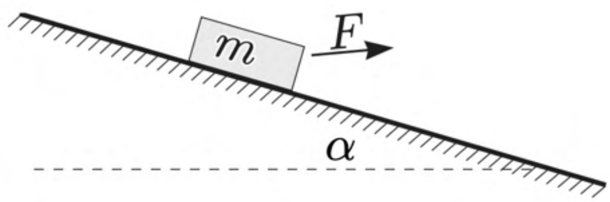

Figure 6: Dislodging block

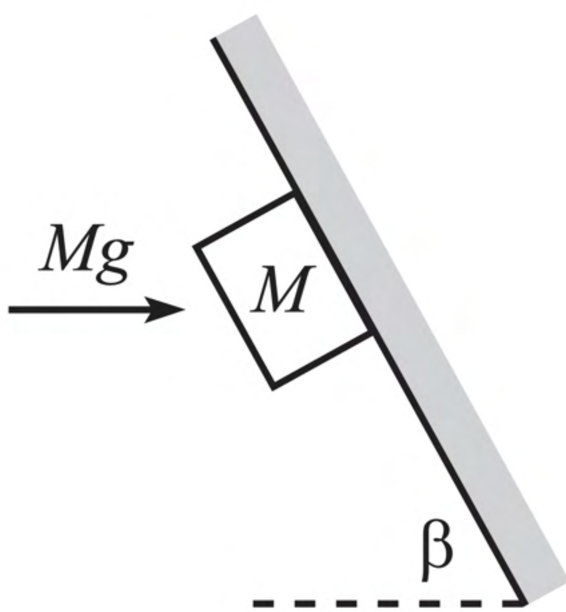

Figure 7: Mass on overhang